GEOMETRY.

DEFINITIONS.

1. If a block of wood or stone be cut in the shape represented in Fig. 1, it will have *six flat faces*.

Each face of the block is called a *surface;* and if these faces are made smooth by polishing, so that, when a straight-edge is applied to any one of them, the straight edge in every part will touch the surface, the faces are called *plane surfaces*, or *planes*.

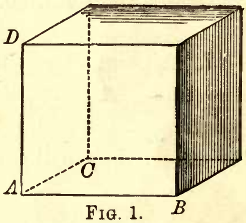


FIG. 1.

2. The *edge* in which any two of these surfaces meet is called a *line*.

3. The *corner* at which any three of these lines meet is called a *point*.

4. For computing its volume, the block is measured in *three* principal directions:

From left to right, A to B.
From front to back, A to C.
From bottom to top, A to D.

These three measurements are called the *dimensions* of the block, and are named *length*, *breadth* (or *width*), *thickness* (*height* or *depth*).

A solid, therefore, has three dimensions, length, breadth, and thickness.

5. The surface of a solid is no part of the solid. It is simply the boundary or limit of the solid. *A surface, therefore, has only two dimensions, length and breadth.* So that, if any number of flat surfaces be put together, they will coincide and form one surface.

6. A line is no part of a surface. It is simply a boundary or limit of the surface. *A line, therefore, has only one dimension, length.* So that, if any number of straight lines be put together, they will coincide and form one line.

7. A point is no part of a line. It is simply the limit of the line. *A point, therefore, has no dimension, but denotes position simply.* So that, if any number of points be put together, they will coincide and form a single point.

8. A *solid*, in common language, is a limited portion of space *filled with matter;* but in Geometry we have nothing to do with the matter of which a body is composed; we study simply its *shape* and *size;* that is, we regard a solid as a limited portion of space which may be occupied by a physical body, or marked out in some other way. Hence,

A geometrical solid is a limited portion of space.

9. It must be distinctly understood at the outset that the points, lines, surfaces, and solids of Geometry are *purely ideal,* though they can be represented to the eye in only a material way. Lines, for example, drawn on paper or on the blackboard, will have some width and some thickness, and will so far fail of being *true lines;* yet, when they are used to help the mind in reasoning, it is assumed that they represent perfect lines, without breadth and without thickness.

10. A point is *represented* to the eye by a fine dot, and named by a letter, as A (Fig. 2); a line is named by two letters, placed one at each end, as BF; a surface is represented and named by the lines which bound it, as $BCDF$; a solid is represented by the faces which bound it.

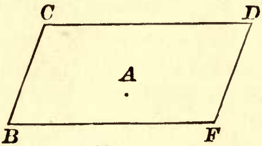


FIG. 2.

11. By supposing a solid to diminish gradually until it vanishes we may consider the vanishing point, a *point* in space, independent of a line, having *position* but *no extent.*

12. If a point moves continuously in space, its path is a line. This line may be supposed to be of *unlimited extent,* and may be considered independent of the idea of a surface.

13. A surface may be conceived as generated by a line moving in space, and as of *unlimited extent.* A surface can then be considered independent of the idea of a solid.

14. A solid may be conceived as generated by a surface in motion.

Thus, in the diagram, let the upright surface $ABCD$ move to the right to the position $EFGH$. The points A, B, C, and D will generate the lines AE, BF, CG, and DH, respectively. The lines AB, BC, CD, and AD will generate the surfaces AF, BG, CH, and AH, respectively. The surface $ABCD$ will generate the solid AG.

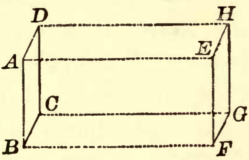


FIG. 3.

15. *Geometry* is the science which treats of *position, form,* and *magnitude.*

16. Points, lines, surfaces, and solids, with their relations, constitute the subject-matter of Geometry.

17. A *straight line*, or *right line*, is a line which has the same direction throughout its whole extent, as the line AB.

18. A *curved line* is a line no part of which is straight, as the line CD.

19. A *broken line* is a series of different successive straight lines, as the line EF.

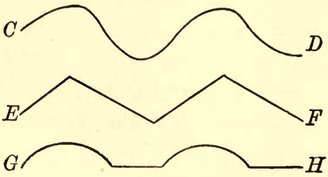


FIG. 4.

20. A *mixed line* is a line composed of straight and curved lines, as the line GH.

A straight line is often called simply a *line*, and a curved line, a *curve*.

21. A *plane surface*, or a *plane*, is a surface in which, if any two points be taken, the straight line joining these points will lie wholly in the surface.

22. A *curved surface* is a surface no part of which is plane.

23. *Figure* or *form* depends upon the relative position of points. Thus, the figure or form of a line (straight or curved) depends upon the relative position of the points in that line; the figure or form of a surface depends upon the relative position of the points in that surface.

24. With reference to *form* or *shape*, lines, surfaces, and solids are called *figures*.

With reference to *extent*, lines, surfaces, and solids are called *magnitudes*.

25. A *plane figure* is a figure all points of which are in the same plane.

26. Plane figures formed by straight lines are called *rectilinear* figures; those formed by curved lines are called *curvilinear* figures; and those formed by straight and curved lines are called *mixtilinear* figures.

27. Figures which have the same *shape* are called *similar* figures. Figures which have the same *size* are called *equivalent* figures. Figures which have the same *shape and size* are called *equal* or *congruent* figures.

28. Geometry is divided into two parts, Plane Geometry and Solid Geometry. Plane Geometry treats of figures all points of which are in the same plane. Solid Geometry treats of figures all points of which are not in the same plane.

Straight Lines.

29. Through a point an indefinite number of straight lines may be drawn. These lines will have different directions.

30. If the direction of a straight line and a point in the line are known, the position of the line is known; in other words, a straight line is *determined* if its direction and one of its points are known. Hence,

All straight lines which pass through the same point in the same direction coincide, and form but one line.

31. Between two points one, and only one, straight line can be drawn; in other words, a straight line is determined if two of the points are known. Hence,

Two straight lines which have two points in common coincide throughout their whole extent, and form but one line.

32. Two straight lines can *intersect* (cut each other) in only one point; for if they had two points common, they would coincide and not intersect.

33. Of all lines joining two points the *shortest* is the straight line, and the length of the straight line is called the *distance* between the two points.

34. A straight line determined by two points is considered as prolonged indefinitely both ways. Such a line is called an *indefinite straight line.*

35. Often only the part of the line between two fixed points is considered. This part is then called a *segment* of the line.

For brevity, we say "the line AB" to designate a segment of a line limited by the points A and B.

36. Sometimes, also, a line is considered as proceeding from a fixed point and extending in only one direction. This fixed point is then called the *origin* of the line.

37. If any point C be taken in a given straight line AB, the two parts CA and CB are said to have *opposite directions* from the point C.

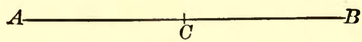


FIG. 5.

38. Every straight line, as AB, may be considered as having opposite directions, namely, from A towards B, which is expressed by saying "line AB"; and from B towards A, which is expressed by saying "line BA."

39. If the magnitude of a given line is changed, it becomes longer or shorter.

Thus (Fig. 5), by prolonging AC to B we add CB to AC, and $AB = AC + CB$. By diminishing AB to C, we subtract CB from AB, and $AC = AB - CB$.

If a given line increases so that it is prolonged by its own magnitude several times in succession, the line is *multiplied*, and the resulting line is called a *multiple* of the given line. Thus (Fig. 6), if $AB = BC = CD = DE$, then $AC = 2\,AB$, $AD = 3\,AB$, and $AE = 4\,AB$. Also, $AB = \frac{1}{2}AC$, $AB = \frac{1}{3}AD$, and $AB = \frac{1}{4}AE$. Hence,

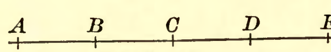


FIG. 6.

44. Two angles are called *adjacent* when they have the same vertex and a common side between them; as, the angles BOD and AOD (Fig. 10).

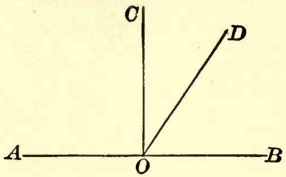


FIG. 10.

45. When one straight line stands upon another straight line and makes the *adjacent angles equal*, each of these angles is called a *right angle*. Thus, the equal angles DCA and DCB (Fig. 11) are each a right angle.

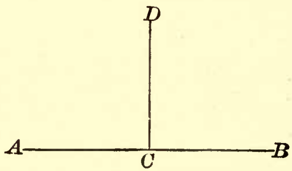


FIG. 11.

46. When the sides of an angle extend in opposite directions, so as to be in the same straight line, the angle is called a *straight angle*. Thus, the angle formed at C (Fig. 11) with its sides CA and CB extending in opposite directions from C, is a straight angle. Hence a *right angle* may be defined as *half a straight angle*.

47. A *perpendicular* to a straight line is a straight line that makes a right angle with it. Thus, if the angle DCA (Fig. 11) is a right angle, DC is perpendicular to AB, and AB is perpendicular to DC.

48. The point (as C, Fig. 11) where a perpendicular meets another line is called the *foot* of the perpendicular.

49. Every angle less than a right angle is called an *acute angle;* as, angle A.

FIG. 12.

50. Every angle greater than a right angle and less than a straight angle is called an *obtuse angle;* as, angle C (Fig. 13).

Maxima and Minima.—Supplementary.

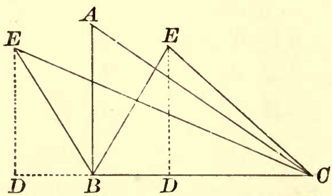
443. Among magnitudes of the same kind, that which is greatest is the *maximum*, and that which is smallest is the *minimum*.

Thus the diameter of a circle is the maximum among all inscribed straight lines; and a perpendicular is the minimum among all straight lines drawn from a point to a given line.

444. *Isoperimetric* figures are figures which have equal perimeters.

Proposition XVIII. Theorem.

445. *Of all triangles having two given sides, that in which these sides include a right angle is the maximum.*



Let the triangles ABC ***and*** EBC ***have the sides*** AB ***and*** BC ***equal respectively to*** EB ***and*** BC***; and let the angle*** ABC ***be a right angle.***

To prove $\triangle ABC > \triangle EBC$.

Proof. From E let fall the $\perp$ ED.

The $\triangle$ ABC and EBC, having the same base BC, are to each other as their altitudes AB and ED. § 370

Now $EB > ED$. § 114

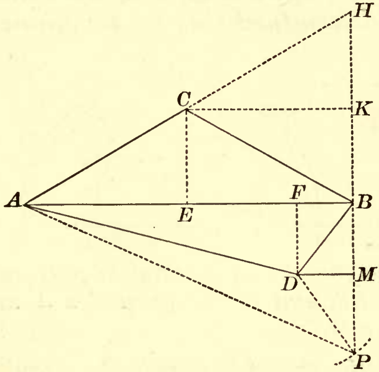
By hypothesis, $EB = AB$.

$\therefore AB > ED$.

$\therefore \triangle ABC > \triangle EBC$. Q. E. D

Proposition XIX. Theorem.

446. *Of all triangles having the same base and equal perimeters, the isosceles triangle is the maximum.*



Let the $\triangle$ ACB and ADB have equal perimeters, and let the $\triangle$ ACB be isosceles.

To prove $\triangle ACB > \triangle ADB$.

Proof. Produce AC to H, making $CH = AC$, and join HB.

ABH is a right angle, for it will be inscribed in the semicircle whose centre is C, and radius CA.

Produce HB, and take $DP = DB$.

Draw CK and $DM \parallel$ to AB, and join AP.

Now $AH = AC + CB = AD + DB = AD + DP$.

But $AD + DP > AP$, hence $AH > AP$.

Therefore $HB > BP$. § 120

But $KB = \frac{1}{2} HB$ and $MB = \frac{1}{2} BP$. § 121

Hence $KB > MB$.

By § 180, $KB = CE$ and $MB = DF$, the altitudes of the $\triangle$ ACB and ADB.

Therefore $\triangle ABC > \triangle ADB$. § 370

Q. E. D.